Module-5: Homeomorphism

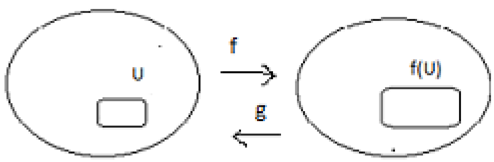
In studying group theory, metric spaces we have observed structure preserving mappings such as isomorphism, isometry. In this module we will discuss structure preserving mappings of topological spaces.

Definition 1. *A continuous map $f : X \to Y$ between topological spaces is said to be a homeomorphism if there exists a continuous map $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$.*

Clearly, here g is actually f^{-1}. So if O is an open set of X then the inverse image of O under f^{-1} is the same as the image of O under the map f. The same thing happens in case of closed sets. So we can define a homeomorphism in the following way.

Theorem 1. *A mapping $f : X \to Y$ between two topological spaces is a homeomorphism if and only if it is continuous and open or closed.*

Proof. Since f is a homeomorphism there exists $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. This means that $f(U) = g^{-1}(U)$ which is open. □



Example 1. *A function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$, where $a \neq 0$ is a homeomorphism.*

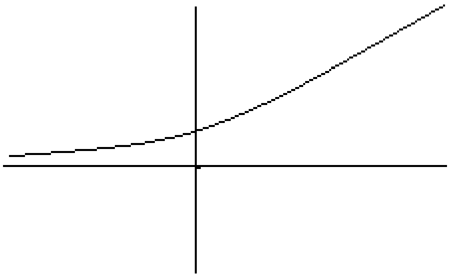
Proof. $g(y) = \frac{f(y)-b}{a}$ is the inverse mapping. Continuity is clear. □

Example 2. *Now we can observe that any two same type of intervals are homeomorphic.*

Without loss of generality let us consider open intervals $(0, 1)$ and $(3, 4)$. The mapping $f : (0, 1) \to (3, 5)$, defined by $f(x) = ax + b$ gives a homeomorphism. We have just to choose a, b suitable real numbers.

Example 3. *This example shows that* $\mathbb{R}$ *and* $(0, \infty)$ *are homeomorphic.*

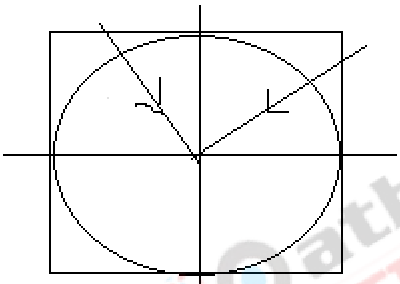
Proof. Let $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$. The following diagram clearly shows that f is a homeomorphism.



□

Example 4. *This example shows that unit square and unit circle are homeomorphic.*

Proof. The following diagram clearly shows the homeomorphism.



□

Example 5. *Consider* $B^n \subset \mathbb{R}^n$ *be the open unit ball, and if we define a map* $f : B^n \to \mathbb{R}^n$ *by*

$$f(x) = \frac{x}{1 - |x|}.$$

Then this gives a homeomorphism from B^n *to* $\mathbb{R}^n$*. An easy computation shows that* $g : \mathbb{R}^n \to B^n$ *defined by*

$$g(x) = \frac{x}{1 + |x|}$$

is the continuous inverse of f*.*

Example 6. *Next we present an example of a homeomorphism between sphere* S^2 *and cube* $C = \{(x, y, z) : \max\{x, y, z\} = 1\}$*. First we define* $: C \to S^2$ *by the mapping*

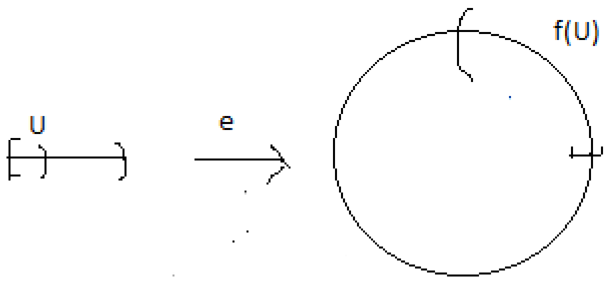
$$f((x, y, z)) = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

Now $g : S^2 \to C$ *defined by*

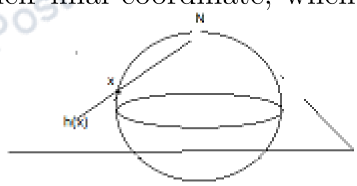
$$g((x, y, z)) = \frac{(x, y, z)}{\max\{|x|, |y|, |z|\}}$$

gives the inverse of f*.*

Example 7. *The mapping $p : [0, 1) \to S^1$ defined by $p(x) = e^{2\pi i x}$ is a continuous bijection, which is not a homeomorphism.*

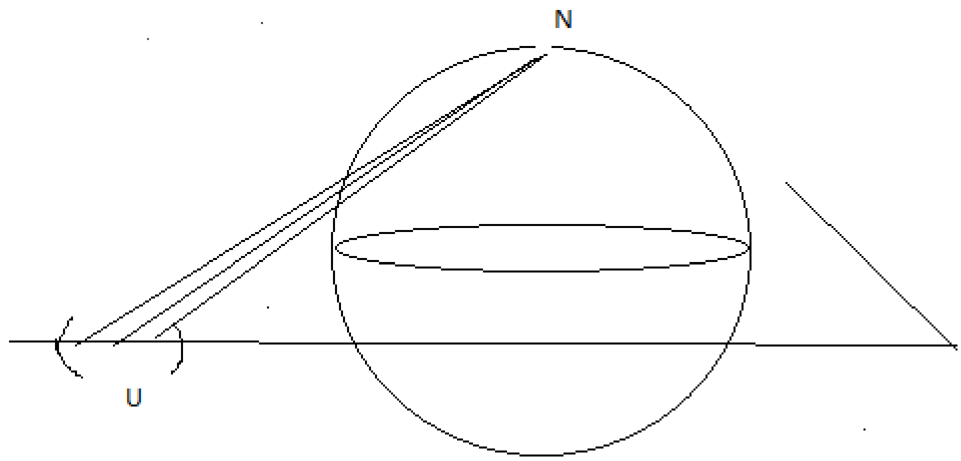


Let S^n denote the n-dimensional sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ taken with the subspace topology. We claim that removing a single point from S^n gives a space homeomorphic to $\mathbb{R}^n$. Which point we remove is irrelevant because we can rotate any point of S^n into any other; for convenience we choose to remove the point $N = (0, 0, \ldots, 0, 1)$. Now the set of points of $\mathbb{R}^{n+1}$ which have zero as their final coordinate, when given the induced



topology, is clearly homeomorphic to $\mathbb{R}^n$.

We define a function $h : S^n \setminus \{N\} \to \mathbb{R}^n$, called stereographic projection, as follows. For any $x \in S^n \setminus \{N\}$, let h(x) be the point of intersection of $\mathbb{R}^n$ and the straight line determined by x and N. Clearly h is bijective. Let O be an open set in $\mathbb{R}^n$, we construct a new set U in S^n whose points are the points of intersection of the straight line segments which start at N and pass through points of O, except the point N(See the diagram in the next page). Then O is open in S^n. But $h^{-1}(O)$ is precisely the set U. Therefore $h^{-1}(O)$ is open in $S^n \setminus \{N\}$. This establishes the continuity of h and a precisely similar argument deals with h^{-1}. Therefore h is a homeomorphism.



Definition 2. *A property say $\mathcal{P}$ of a topological space is said to be topological property if when ever two topological spaces X and Y are homeomorphic and one posses the property then the other will posses the property.*

Example 8. *Let $X = \mathbb{R}$ and let d be the usual metric on $\mathbb{R}$. Let $Y = (0, 1)$ (the open interval) and let ρ be the usual metric on $(0, 1)$. Then X and Y are homeomorphic as topological spaces, but (X, d) is complete and (Y, ρ) is not. So the completeness is not a topological property.*

Theorem 2. *Both T_1 ness and Hausdorffness are topological property.*

Proof. Let X and Y be two topological spaces, $f : X \to Y$ be a homeomorphism and X be T_1. Now if F be a finite set of Y then $|f^{-1}(F)| = |F|$ and hence $f^{-1}(F)$ is closed and f being closed $F = f(f^{-1}(F))$ is also closed. Hence Y is T_1.

Let $f : X \to Y$ be a homeomorphism, X be Hausdorff and let $x \neq y$ be two distinct points in Y. Choose u and v be unique preimages of x and y respectively. Then there exist disjoint open sets U and V in X containing x and y respectively. Then using openness of f we get disjoint open sets $f(U)$ and $f(V)$ containing $x \neq y$ respectively. This proves that proved that Y is Hausdorff. □

The following result will be needed in the study of manifold. By a disc we shall mean any space homeomorphic to the closed unit disc D in $\mathbb{R}^2$. If A is a disc, and if $h : A \to D$ is a homeomorphism, then $h^{-1}(S^1)$ is called the boundary of A and is written ∂A.

Theorem 3. *Any homeomorphism from the boundary of a disc to itself can be extended to a homeomorphism of the whole disc.*

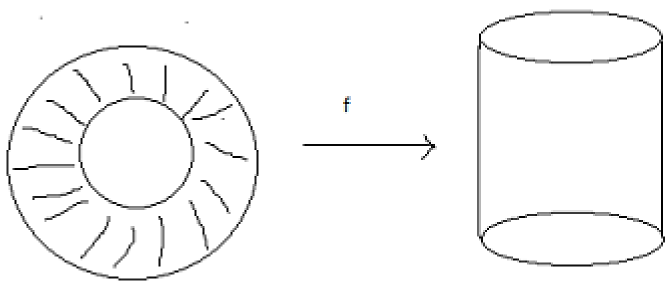
Proof. Let A be a disc and choose a homeomorphism $h : A \to D$. Given a homeomorphism $g : \partial A \to \partial A$ we can easily extend $hgh^{-1} : S^1 \to S^1$ to a homeomorphism of all of D as follows. Send 0 to 0, and if $x \in D \setminus \{0\}$ send x to the point $\|x\|hgh^{-1}\left(\frac{x}{\|x\|}\right)$. In other words extend conically. If we call this extension f, then $h^{-1}fh$ extends g to a homeomorphism of all of A as required.

□

We have already discussed about finite product topology. One can observe easily that each projection map is continuous. Let us examine the following example.

Example 9. *If we view points in the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$ as angles θ, then polar coordinates give a homeomorphism $f : \mathbb{S}^1 \times (0, \infty) \to \mathbb{R}^2 \setminus \{0\}$ defined by $f(\theta, r) = (r cos\theta, r sin\theta)$. This is one-to-one and onto since each point in $\mathbb{R}^2$, other than the origin has unique polar coordinates (θ, r). To see that f is a homeomorphism, just observe that it takes a basic open set $U \times V$, (where U is an open interval (θ_0, θ_1) and V is an open interval (r_0, r_1)) to an open polar rectangle and such rectangles form a basis for the topology on $\mathbb{R}^2 \setminus \{0\}$, as a subspace of $\mathbb{R}^2$. By restricting f to a product $\mathbb{S}^1 \times [a, b]$ for $0 < a < b$ we obtain a homeomorphism from this product to a closed annulus in $\mathbb{R}^2$, the region between two concentric circles.*

Example 10. *A product $\mathbb{S}^1 \times [1, 2]$ is homeomorphic to a cylinder as well as to an annulus. If we use cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$ then a cylinder is specified by taking r to be a constant 1, letting range over the circle $\mathbb{S}^1$, and restricting z to an interval $[1, 2]$.*



Consider the spaces Consider $[0, 1]$ and $\mathbb{R}$. This following Theorem shows that we cann't hope a Theorem like Cantor Bernstein Theorem.